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Note

A Note about a Theorem by Maamoun on Decompositions of Digraphs into Elementary Directed Paths or Cycles

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Maamoun (*J. Combin. Theory Ser. B* **38** (1985), 97–101) has found an upper bound on the minimum number of elementary directed paths or elementary directed cycles which can partition the arcs of a digraph G. We slightly improve his theorem by specifying the number of elementary directed paths needed in such a partition.

1. Introduction

Following a conjecture by Bienia and myself, Maamoun [2] has proved that there exists an elementary directed path or an elementary directed cycle which meets every demi-cocycle of maximum size. As a consequence, he has shown that there exists a partition into λ elementary directed paths or elementary directed cycles, where λ denotes the maximum size of a demi-cocycle. We show that we can find a partition with at most λ elementary directed paths or elementary directed cycles and with exactly $\sum_x \max(d_G^+(x) - d_G^-(x), 0)$ elementary directed paths, where $d_G^+(x)$ (resp. $d_G^-(x)$) denotes the outdegree (resp. the indegree) of a vertex x.

2. Definitions and Notations

General definitions are classical, see [1]. Digraphs G considered here may have multiple arcs, but loops are not allowed. We only consider directed elementary paths or directed elementary cycles. From now on we shall omit the word "elementary." A demi-cocycle is the set of all the arcs going

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from a subset A to $V\backslash A$. It is denoted by $\omega_G^+(A)$. λ_G will be $\sup_{A\subset V}|\omega_G^+(A)|$. A pseudo-source is a vertex x with $d_G^+(x)>d_G^-(x)$ and a pseudo-sink is a vertex x with $d_G^-(x)>d_G^+(x)$.

3. The Results

Theorem 1. *Let G be a digraph. Then at least one of the following properties holds*:

(a) *There exists a directed cycle meeting all the demi-cocycles of size* λ_G.

(b) *Every directed path of maximum length has for initial endpoint a pseudo-source and for terminal endpoint a pseudo-sink, and meets every demi-cocycle of maximum size.*

Proof. Maamoun has proved the above statement almost entirely but he has not specified the properties of the endpoints of the considered directed paths. So, all we have to do is to prove that if statement (a) does not hold, a path of maximum length has for initial endpoint a pseudo source and for terminal endpoint a pseudo-sink.

For this purpose, let us consider a directed path of maximum length and let $P=(x_0, x_1,..., x_m)$ be this path. There exists at least one vertex x with $x\in P\cap\Gamma_G^-(x_0)$, otherwise P would not be of maximum size or x_0 would be a pseudo-source.

Let x_i be the vertex belonging to $P\cap\Gamma_G^-(x_0)$ such that $x_k\notin\Gamma_G^-(x_0)$ $(k>i)$. The directed cycle $C=(x_0, x_1,..., x_i, x_0)$ does not meet ever demi-cocycle of maximum size. So there is a demi-cocycle, let us say $\omega_G^+(A)$, which is not met. Hence C is contained in A or in $V\backslash A$.

First of all, C cannot be contained in $V\backslash A$. Indeed we have $\Gamma_G^-(x_0)\subset C$ (as P is of maximum length). Hence $\Gamma_G^-(x_0)$ is contained in $V\backslash A$ but we have also $x_1\in\Gamma_G^+(x_0)\cap V-A$. Hence we would have $|\omega_G^+(x_0\cup A)|>|\omega_G^+(A)|$. A contradiction.

If C is contained in A, for the same reasons as above, we have $|\omega_G^+(A\backslash x_0)|>|\omega_G^+(A)|$ unless $d_G^+(x_0)>d_G^-(x_0)$ and we have proved our assertion.

The same reasonning shows that x_m must be a pseudo-sink and Theorem 1 is proved.

Theorem 2. *Let G be a digraph. It is possible to partition the arcs of G into at most* λ_G *directed paths or directed cycles with exactly* $\sum_{x\in V}\max(d_G^+(x)-d_G^-(x), 0)$ *directed paths in the partition.*

The proof is easy to derive from Theorem 1, and is left to the reader.

REFERENCES

1. C. BERGE, "Graphs and Hypergraphs," North-Holland, Amsterdam, 1973.
2. M. MAAMOUN, Decompositions of digraphs into paths and cycles, *J. Combin. Theory Ser. B* **38** (1985), 97–101.